A NOTE ON THE "LOGARITHMIC-$\mathscr{W}_3$" OCTUPLET ALGEBRA AND ITS NICHOLS ALGEBRA

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ABSTRACT. We describe a Nichols-algebra-motivated construction of an octuplet chiral algebra that is a "$\mathscr{W}_3$-counterpart" of the triplet algebra of $(p,1)$ logarithmic models of two-dimensional conformal field theory.

1. INTRODUCTION

Logarithmic models of two-dimensional conformal field theory can be defined as centralizers of Nichols algebras [1, 2]. For this, the generators F_i of a given Nichols algebra $\mathfrak{B}(X)$ with diagonal braiding [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] are to be realized as

$$F_i = \oint e^{\alpha_i.\varphi}, \quad 1 \leqslant i \leqslant \text{rank} \equiv \theta,$$

where $\varphi(z)$ is a θ-plet of scalar fields and $\alpha_i \in \mathbb{C}^\theta$ are chosen so as to reproduce the given braiding coefficients $q_{i,j}$ in

$$\Psi : F_i \otimes F_j \mapsto q_{i,j} F_j \otimes F_i, \quad 1 \leqslant i, j \leqslant \theta.$$

The coefficients are standardly arranged into a braiding matrix $(q_{i,j})_{\substack{1\leqslant i\leqslant \text{rank}\\ 1\leqslant j\leqslant \text{rank}}}$. The relation between the braiding matrix and the screening momenta is postulated [2] in the form of equations

$$q_{j,j} = e^{i\pi\alpha_j.\alpha_j}, \qquad q_{j,k}q_{k,j} = e^{2i\pi\alpha_j.\alpha_k}$$

and the logical-"or" conditions

$$a_{i,j}\alpha_i.\alpha_i = 2\alpha_i.\alpha_j, \quad \bigvee \quad (1-a_{i,j})\alpha_i.\alpha_i = 2$$

imposed for each pair $i \neq j$ and involving the Cartan matrix $a_{i,j}$ associated with the given braiding matrix (see, e.g., [18] and the references therein).

In this note, we describe some details related to the construction of the octuplet algebra [2] that can be considered a "logarithmic extension" of the $\mathscr{W}_3$ algebra [19] similarly to how the triplet algebra [21, 22, 23] is a "logarithmic extension" of the Virasoro algebra. The starting point is a particular item in Heckenberger's list of rank-2 Nichols algebras with diagonal braiding (which is item **5.7**(1) in [20])—the braiding matrix

$$q_{ij} = \begin{pmatrix} \mathfrak{q}^2 & \mathfrak{q}^{-1} \\ \mathfrak{q}^{-1} & \mathfrak{q}^2 \end{pmatrix}, \tag{1.1}$$

where $\mathfrak{q}^2$ is a primitive $2p$th root of unity. We choose

$$\mathfrak{q} = e^{\frac{i\pi}{p}} \tag{1.2}$$

with $p = 2, 3, \ldots$. This choice leads to $(p,1)$-type logarithmic CFT models [21, 22, 23, 24, 25, 26, 27], in contrast to (p,p') models that follow if $\mathfrak{q}$ is chosen as $e^{\frac{i\pi p'}{p}}$ instead. The main expectation associated with $(p,1)$-type models is that their representation categories are "*very* closely related" [25, 28, 29] to an appropriate representation category on the algebraic side, which in the braided case is some category of Yetter–Drinfeld $\mathfrak{B}(X)$-modules (cf. [30]). In this paper, we therefore proceed along two routes: (i) describing the structure of the $\mathfrak{B}(X)$ algebra associated with (1.1) (solely with the choice in (1.2)) and its suitable Yetter–Drinfeld modules, and (ii) discussing some properties of the octuplet algebra that centralizes this $\mathfrak{B}(X)$. None of the two directions is pursued to the point where they actually meet (which would mean constructing a functor), but the results presented here hopefully bring us somewhat closer to that point.

2. The Nichols algebra

2.1. Presentation for $\mathfrak{B}(X)$. We first recall the presentation of the relevant Nichols algebra, as a quotient of the tensor algebra. Our starting point is a two-dimensional braided vector space X with the preferred basis F_1, F_2 and the above braiding matrix in this basis. The Nichols algebra $\mathfrak{B}(X)$ is the quotient by a graded ideal $\mathfrak{I}$ [16, 20],

$$\mathfrak{B}(X) = T(X)/\big([F_1,[F_1,F_2]],\ [F_2,[F_2,F_1]],\ F_1^p,\ [F_2,F_1]^p,\ F_2^p\big), \quad \dim\mathfrak{B}(X) = p^3, \tag{2.1}$$

If $p=2$, the double-bracket generators of the ideal are absent. The brackets here denote $\mathfrak{q}$-commutators determined by the braiding matrix: $[F_1,F_2] = F_1F_2 - \mathfrak{q}^{-1}F_2F_1$, $[F_2,F_1] = F_2F_1 - \mathfrak{q}^{-1}F_1F_2$, and so on by multiplicativity of the "q"-factor, whence the two double commutators in the ideal are explicitly given by

$$[F_1,[F_1,F_2]] = F_1^2F_2 - (\mathfrak{q}+\mathfrak{q}^{-1})F_1F_2F_1 + F_2F_1^2,$$
$$[F_2,[F_2,F_1]] = F_2^2F_1 - (\mathfrak{q}+\mathfrak{q}^{-1})F_2F_1F_2 + F_1F_2^2.$$

A PBW basis in $\mathfrak{B}(X)$ is given by $F_1^rF_3^tF_2^s$, $0 \leqslant r,s,t \leqslant p-1$ [20], where

$$F_3 = [F_2,F_1].$$

The double-bracket relations in the ideal can also be rewritten as $F_2F_3 = \mathfrak{q}F_3F_2$ and $F_3F_1 = \mathfrak{q}F_1F_3$.

Multiplication in $\mathfrak{B}(X) = T(X)/\mathfrak{I}$ is the one induced by "concatenation" in the tensor algebra, $X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes(m+n)}$, $(x_1,\ldots,x_m)\otimes(y_1,\ldots,y_n) \mapsto (x_1,\ldots,x_m,y_1,\ldots,y_n)$. It is then relatively straightforward to show that the multiplication table of the PBW basis elements is

$$(F_1^{r_1}F_3^{t_1}F_2^{s_1})(F_1^{r_2}F_3^{t_2}F_2^{s_2}) = \sum_{i=0}^{\min(s_1,r_2)} \mathfrak{q}^{t_1(r_2-i)+t_2(s_1-i)-s_1r_2+i(1+i)/2}\langle i\rangle! \left\langle{s_1 \atop i}\right\rangle \left\langle{r_2 \atop i}\right\rangle F_1^{r_1+r_2-i}F_3^{t_1+t_2+i}F_2^{s_1+s_2-i}. \tag{2.2}$$

Comultiplication is by "deshuffling," determined by the defining property of a braided Hopf algebra and the fact that F_1 and F_2 are primitive.

2.2. $\mathfrak{B}(X)$ as a subalgebra in $T(X)$. For any Nichols algebra $\mathfrak{B}(X)$, the graded ideal $\mathfrak{I}$ such that $\mathfrak{B}(X) = T(X)/\mathfrak{I}$ is known to be the kernel of the total braided symmetrizer map in each grade, $\mathfrak{S}_n : X^{\otimes n} \to X^{\otimes n}$. Mapping by $\mathfrak{S}_n$ in each grade therefore results in an equivalent description of $\mathfrak{B}(X)$ with multiplication given by the shuffle product

$$* : (x_1,\dots,x_m)\otimes(y_1,\dots,y_n) \mapsto \text{Ш}_{m,n}(x_1,\dots,x_m,y_1,\dots,y_n),$$

and comultiplication by deconcatenation (see [1] for the definition of shuffles and the braided symmetrizer; the only notational difference is that $*$ is not used for the shuffle product there).

We let $\mathrm{B}(r,t,s)$ be the image of $F_1^rF_3^tF_2^s$ under the map by the braided symmetrizer, or more precisely,

$$\mathrm{B}(r,t,s) = \frac{1}{\langle r\rangle!\,\langle s\rangle!\,\langle t\rangle!\,(1-\mathfrak{q}^2)^t}\mathfrak{S}_{r+2t+s}(F_1^rF_3^tF_2^s).$$

In particular,

$$\begin{aligned}
&\mathrm{B}(1,0,0) = F_1, &&\mathrm{B}(2,0,0) = F_1F_1,\\
&\mathrm{B}(0,0,1) = F_2, &&\mathrm{B}(1,0,1) = F_1F_2 + \mathfrak{q}^{-1}F_2F_1,\\
& &&\mathrm{B}(0,0,2) = F_2F_2,\\
& &&\mathrm{B}(0,1,0) = -\mathfrak{q}^{-2}F_2F_1.
\end{aligned}$$

2.2.1. The shuffle product of $\mathrm{B}(r_1,t_1,s_1)$ and $\mathrm{B}(r_2,t_2,s_2)$ follows from (2.2):

$$\begin{aligned}\mathrm{B}(r_1,t_1,s_1) * \mathrm{B}(r_2,t_2,s_2) = {}&\sum_{i=0}^{\min(s_1,r_2)} \left\langle{r_1+r_2-i \atop r_1}\right\rangle \left\langle{s_1+s_2-i \atop s_2}\right\rangle \frac{(1-\mathfrak{q}^2)^i\langle t_1+t_2+i\rangle!}{\langle t_1\rangle!\,\langle t_2\rangle!\,\langle i\rangle!}\,\mathfrak{q}^{t_1(r_2-i)+t_2(s_1-i)-s_1r_2+i(i+1)/2}\\
&\times \mathrm{B}(r_1+r_2-i,t_1+t_2+i,s_1+s_2-i).\end{aligned} \tag{2.3}$$

and the coproduct is

$$\begin{aligned}\Delta : \mathrm{B}(r,t,s) \mapsto {}&\sum_{j=0}^{r}\sum_{m=0}^{s}\sum_{k=0}^{t}\sum_{i=0}^{k}(-1)^i\mathfrak{q}^{-i(i+3)/2+(k-m-2i)j+m(t-i-k)}\\
&\times \left\langle{i+j \atop i}\right\rangle \left\langle{i+m \atop i}\right\rangle \langle i\rangle!\,\mathrm{B}(r-j,k-i,i+m)\otimes \mathrm{B}(j+i,t-k,s-m),\end{aligned} \tag{2.4}$$

where terms with the lowest grades in the first tensor factor are

$$\begin{aligned}&= 1\otimes \mathrm{B}(r,t,s) + F_1\otimes \mathrm{B}(r-1,t,s)\\ &\quad + \mathfrak{q}^{t-r}F_2\otimes \mathrm{B}(r,t,s-1) - \mathfrak{q}^{-r-2}\langle r+1\rangle F_2\otimes \mathrm{B}(r+1,t-1,s)\\ &\quad + \dots\end{aligned}$$

(the dots stand for terms $\mathrm{B}(r',t',s')\otimes \mathrm{B}(r'',t'',s'')$ with $r'+2t'+s'\geqslant 2$).

2.2.2. Remark. Although this is obvious, we note explicitly that the "Serre relations"—the double q-commutators in the ideal—are resolved in terms of the shuffle product in the sense that the relations

$$\begin{aligned}F_1 * F_1 * F_2 - (\mathfrak{q}+\mathfrak{q}^{-1})F_1 * F_2 * F_1 + F_2 * F_1 * F_1 = 0,\\ F_2 * F_2 * F_1 - (\mathfrak{q}+\mathfrak{q}^{-1})F_2 * F_1 * F_2 + F_1 * F_2 * F_2 = 0\end{aligned}$$

hold identically for the shuffle product defined by the braiding matrix (1.1).

2.2.3. The action of the antipode on the PBW basis elements is defined by the formulas

$$\begin{aligned}S(\mathrm{B}(r,0,0)) &= (-1)^r\mathfrak{q}^{r(r-1)}\mathrm{B}(r,0,0),\\ S(\mathrm{B}(0,t,0)) &= \sum_{i=0}^{t}(-1)^t\mathfrak{q}^{\frac{1}{2}i(i-1)-(i+3)t+t^2}\langle i\rangle!\ \mathrm{B}(i,t-i,i),\\ S(\mathrm{B}(0,0,s)) &= (-1)^s\mathfrak{q}^{s(s-1)}\mathrm{B}(0,0,s)\end{aligned}$$

and by the fact that S is a braided antiautomorphism:

$$S(\mathrm{B}(r,t,s)) = \mathfrak{q}^{rt-rs+ts}S(\mathrm{B}(0,0,s)) * S(\mathrm{B}(0,t,0)) * S(\mathrm{B}(r,0,0)).$$

2.3. Vertex operators and Yetter–Drinfeld $\mathfrak{B}(X)$ modules. Multivertex $\mathfrak{B}(X)$ module comodules, which are Yetter–Drinfeld modules, were defined in [1]. We here realize simple Yetter–Drinfeld modules of our $\mathfrak{B}(X)$ in terms of one-vertex modules.

2.3.1. The Y spaces. Let $Y^{\{n_1,n_2\}}$ be a one-dimensional vector space with basis $V^{\{n_1,n_2\}}$ and braiding $\psi : \mathfrak{B}(X)\otimes Y^{\{n_1,n_2\}} \to Y^{\{n_1,n_2\}}\otimes \mathfrak{B}(X)$ and $Y^{\{n_1,n_2\}}\otimes \mathfrak{B}(X) \to \mathfrak{B}(X)\otimes Y^{\{n_1,n_2\}}$ defined by

$$\begin{aligned}\psi(F_i\otimes V^{\{n_1,n_2\}}) &= \mathfrak{q}^{1-n_i}V^{\{n_1,n_2\}}\otimes F_i,\\ \psi(V^{\{n_1,n_2\}}\otimes F_i) &= \mathfrak{q}^{1-n_i}F_i\otimes V^{\{n_1,n_2\}},\end{aligned}$$

$i=1,2$. Every space $\mathfrak{B}(X)\otimes V^{\{n_1^1,n_2^1\}}\otimes \mathfrak{B}(X)\otimes V^{\{n_1^2,n_2^2\}}\otimes\ldots\otimes V^{\{n_1^N,n_2^N\}}$ is a Yetter–Drinfeld $\mathfrak{B}(X)$ module. Taking the a_i^j to be generic leads to continuum families of such modules, leaving us with no chance of a nice correspondence with any type of "reasonably rational" CFT model. The choice of the possible a_i^j values is governed by the requirement that all of them (and the braided vector space X itself) be objects of a suitable ${}^{H}_{H}\mathscr{YD}$

category of Yetter–Drinfeld modules over a nonbraided Hopf algebra H. In the case of diagonal braiding, more specifically, $H = k\Gamma$ for an Abelian group Γ, which can then be considered the origin of the appropriate discreteness in the a_i^j values. We do not pursue this line in this paper, and simply assume that the a_i^j take integer values.

We consider one-vertex modules $\mathfrak{B}(X)\otimes V^{\{n_1,n_2\}}$ and for brevity write

$$\mathrm{B}(r,t,s)^{\{n_1,n_2\}} = \mathrm{B}(r,t,s)\otimes V^{\{n_1,n_2\}} \in \mathfrak{B}(X)\otimes Y^{\{n_1,n_2\}},$$

and, in particular,

$$F_i^{\{n_1,n_2\}} = F_i\otimes V^{\{n_1,n_2\}} \in \mathfrak{B}(X)\otimes Y^{\{n_1,n_2\}}$$

(but $\mathrm{B}(0,0,0)^{\{n_1,n_2\}} = 1\otimes V^{\{n_1,n_2\}}$ is normally written as $V^{\{n_1,n_2\}}$).

2.3.2. Left adjoint action. The formulas for the product, coproduct, and antipode in **2.2.1–2.2.3** allow calculating the left adjoint action of the $\mathfrak{B}(X)$ generators on one-vertex modules:

$$\begin{aligned} F_1 \blacktriangleright \mathrm{B}(r,t,s)^{\{n_1,n_2\}} &= \langle r+1\rangle(1-\mathfrak{q}^{2(r-s+t+1-n_1)})\mathrm{B}(r+1,t,s)^{\{n_1,n_2\}} \\ &\quad - \mathfrak{q}^{2r-2s+t-2n_1+3}\langle t+1\rangle(1-\mathfrak{q}^2)\mathrm{B}(r,t+1,s-1)^{\{n_1,n_2\}} \end{aligned}$$

and

$$\begin{aligned} F_2 \blacktriangleright \mathrm{B}(r,t,s)^{\{n_1,n_2\}} &= \mathfrak{q}^{1-r}\langle t+1\rangle(1-\mathfrak{q}^2)\mathrm{B}(r-1,t+1,s)^{\{n_1,n_2\}} \\ &\quad + \mathfrak{q}^{t-r}\langle s+1\rangle(1-\mathfrak{q}^{2(s+1-n_2)})\mathrm{B}(r,t,s+1)^{\{n_1,n_2\}}. \end{aligned}$$

These formulas depend on n_1 and n_2 only through $(a_i \bmod p)$. The $\mathfrak{B}(X)$ coaction is given by literally applying formula (2.4) to $\mathrm{B}(r,t,s)\otimes V^{\{n_1,n_2\}}$ (and is entirely independent of a_i).

2.3.3. Simple Yetter–Drinfeld modules. A simple Yetter–Drinfeld $\mathfrak{B}(X)$-module $\mathscr{Y}_{n_1,n_2}$ is generated from $V^{\{n_1,n_2\}}$ under the action of $\mathfrak{B}(X)$; its dimension is given by

$$d(p,n_1,n_2) = \begin{cases} d(\overline{n_1},\overline{n_2}), & \overline{n_1}+\overline{n_2} \leqslant p, \\ d(\overline{n_1},\overline{n_2}) - d(p-\overline{n_1},p-\overline{n_2}), & \overline{n_1}+\overline{n_2} \geqslant p+1, \end{cases}$$

where $d(n_1,n_2) = \frac{1}{2}n_1 n_2 (n_1+n_2)$ and

$$\overline{x} = \begin{cases} p, & (x \bmod p) = 0, \\ x \bmod p, & \text{otherwise.} \end{cases}$$

3. The octuplet algebra centralizing $\mathfrak{B}(X)$

We next discuss a CFT construction related to our $\mathfrak{B}(X)$.

3.1. Screenings and their zero-momentum centralizer. We identify the $\mathfrak{B}(X)$ generators with two screenings

$$F_\alpha = F_1 = \oint e^{\varphi_\alpha}, \qquad F_\beta = F_2 = \oint e^{\varphi_\beta}, \tag{3.1}$$

where $\varphi_\alpha(z)$ and $\varphi_\beta(z)$ are two scalar fields whose OPEs are defined in accordance with the braiding matrix as follows:

$$\varphi_\alpha(z)\,\varphi_\alpha(w) = \frac{2}{p}\log(z-w), \quad \varphi_\alpha(z)\varphi_\beta(w) = -\frac{1}{p}\log(z-w),$$
$$\varphi_\beta(z)\varphi_\beta(w) = \frac{2}{p}\log(z-w).$$

It follows from the formulas in [2] that the centralizer (“kernel”) of screenings (3.1) contains a Virasoro algebra with the central charge

$$c = 50 - \frac{24}{p} - 24p = -\frac{2(3p-4)(4p-3)}{p}. \tag{3.2}$$

This Virasoro algebra is represented by the energy–momentum tensor

$$T(z) = \tfrac{p}{3}\partial\varphi_\alpha\partial\varphi_\alpha(z) + \tfrac{p}{3}\partial\varphi_\alpha\partial\varphi_\beta(z) + \tfrac{p}{3}\partial\varphi_\beta\partial\varphi_\beta(z) - (p-1)\partial^2\varphi_\alpha(z) - (p-1)\partial^2\varphi_\beta(z).$$

In addition to the Virasoro algebra, the kernel of the screenings contains the dimension-3 Virasoro primary field (omitting the conventional (z) arguments of fields)

$$\begin{aligned} W(z) &= \partial\varphi_\alpha\partial\varphi_\alpha\partial\varphi_\alpha + \frac{3}{2}\partial\varphi_\alpha\partial\varphi_\alpha\partial\varphi_\beta - \frac{3}{2}\partial\varphi_\alpha\partial\varphi_\beta\partial\varphi_\beta - \partial\varphi_\beta\partial\varphi_\beta\partial\varphi_\beta \\ &- \frac{9(p-1)}{2p}\partial^2\varphi_\alpha\partial\varphi_\alpha - \frac{9(p-1)}{4p}\partial^2\varphi_\alpha\partial\varphi_\beta + \frac{9(p-1)}{4p}\partial^2\varphi_\beta\partial\varphi_\alpha + \frac{9(p-1)}{2p}\partial^2\varphi_\beta\partial\varphi_\beta \\ &+ \frac{9(p-1)^2}{4p^2}\partial^3\varphi_\alpha - \frac{9(p-1)^2}{4p^2}\partial^3\varphi_\beta. \end{aligned} \tag{3.3}$$

The operator product of this field with itself is given by

$$\begin{aligned} W(z)W(w) &= \frac{81(3p-5)(3p-4)(4p-3)(5p-3)}{4p^5}\frac{1}{(z-w)^6} - \frac{243}{4p^4}\frac{(3p-5)(5p-3)T(w)}{(z-w)^4} \\ &- \frac{243}{8p^4}\frac{(3p-5)(5p-3)\partial T(w)}{(z-w)^3} + \frac{243}{16p^4}\frac{8pTT(w) - 9(p-1)^2\partial^2 T(w)}{(z-w)^2} \\ &+ \frac{243}{8p^4}\frac{4p(\partial T)T(w) - (p-1)^2\partial^3 T(w)}{z-w}, \end{aligned}$$

where $TT(w)$ is the normal-ordered product $T(w)T(w)$ (and similarly for $(\partial T)T(w)$). This OPE defines the $\mathscr{W}_3$ algebra [19] (also see [31]).

In an equivalent description, the $\mathscr{W}_3$ algebra relations for the modes introduced as $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ and $W(z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-3}$ are

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(50 - \frac{24}{p} - 24p)(m-1)m(m+1)\delta_{m+n,0},$$
$$[L_m, W_n] = (2m-n)W_{m+n},$$
$$[W_m, W_n] = -\frac{81(3p-5)(5p-3)}{8p^4}(m-n)\Big(\frac{(m+n+3)(m+n+2)}{5} - \frac{(m+2)(n+2)}{2}\Big)L_{m+n}$$

$$+\frac{243}{4p^3}(m-n)\Lambda_{m+n}+\frac{27(3p-5)(3p-4)(4p-3)(5p-3)}{160p^5}m(m^2-1)(m^2-4)\delta_{m+n,0},$$

where

$$\Lambda_m=\sum_{n\leqslant-2}L_nL_{m-n}+\sum_{n\geqslant-1}L_{m-n}L_n-\frac{3}{10}(m+3)(m+2)L_m.$$

3.2. Long screenings. The $\mathscr{W}_3$ algebra is also centralized by two “long” screenings

$$\mathcal{E}_\alpha=\oint e^{-p\varphi_\alpha}\quad\text{and}\quad\mathcal{E}_\beta=\oint e^{-p\varphi_\beta}. \tag{3.4}$$

Because

$$[F_i,\mathcal{E}_j]=0,$$

the long screenings act on the kernel of the F_α and F_β, and are therefore a useful tool in studying that kernel.

3.3. Remark. We note that, generally, given the screenings $F_i=\oint e^{\varphi_i}=\oint e^{\alpha_i\cdot\varphi}$, $i=1,\dots,\theta$, the Virasoro dimension of a vertex $e^{\mu\cdot\varphi(z)}$ with $\mu=\sum_{i=1}^{\theta}c_i\alpha_i$ is

$$\Delta(c)=\sum_{i=1}^{\theta}c_i\big(1-\frac{\alpha_i.\alpha_i}{2}\big)+\frac{1}{2}\sum_{i,j=1}^{\theta}c_ic_j\,\alpha_i.\alpha_j.$$

We list the generators of the ideal in (2.1) together with the vertex operators that naively (by momentum counting) correspond to them, and with the Virasoro dimensions of these vertices:

$$\begin{array}{lllll}
[F_1,[F_1,F_2]], & [F_2,[F_2,F_1]], & F_1^p, & [F_1,F_2]^p, & F_2^p,\\
e^{2\varphi_\alpha(z)+\varphi_\beta(z)}, & e^{\varphi_\alpha(z)+2\varphi_\beta(z)}, & e^{p\varphi_\alpha(z)}, & e^{p\varphi_\alpha(z)+p\varphi_\beta(z)}, & e^{p\varphi_\beta(z)},\\
3, & 3, & 2p-1, & 3p-2, & 2p-1.
\end{array} \tag{3.5}$$

3.4. The octuplet algebra. The field

$$\mathcal{W}(z)=e^{p\varphi_\alpha(z)+p\varphi_\beta(z)},$$

which is the top-dimension field in (3.5), is in the kernel of F_α and F_β and is a $\mathscr{W}_3$-primary field of dimension $\Delta=3p-2$ and the W_0 eigenvalue zero. To describe how it is mapped by the long screenings, we need a reminder on $\mathscr{W}_3$ singular vectors.

3.4.1. Singular vectors in $\mathscr{W}_3$ Verma modules. We recall from [32] (also see [31] and the references therein) that highest-weight vectors of the $\mathscr{W}_3$ algebra can be conveniently parameterized by (x,y) such that

$$\begin{aligned}
L_m\big|x,y\big\rangle&=0,\quad m\geqslant1,\\
W_m\big|x,y\big\rangle&=0,\quad m\geqslant1,\\
L_0\big|x,y\big\rangle&=\Big(\frac{x^2+y^2+xy}{3}-\frac{(p-1)^2}{p}\Big)\big|x,y\big\rangle,
\end{aligned}$$

$$W_0\big|x,y\big\rangle = \frac{1}{2p^{3/2}}(x-y)(2x+y)(x+2y)\big|x,y\big\rangle.$$

The two numbers x and y are defined not uniquely but up to a Weyl transformation; the Weyl group orbit of (x,y) also contains $(-x,x+y)$, $(x+y,-y)$, $(y,-x-y)$, $(-x-y,x)$, and $(-y,-x)$. We write $\mathcal{V}(z) \doteq \big|x,y\big\rangle$ for any field/state $\mathcal{V}(z)$ that satisfies the above conditions.

In what follows, we use the conditions for the existence of singular vectors in Verma modules of the $\mathscr{W}_3$ algebra [33, 34, 32]. Whenever a state can be represented as $\big|x,y\big\rangle$ with $x = a\sqrt{p} - \frac{c}{\sqrt{p}}$ for integer a and c such that $ac > 0$, there is a singular vector on the level ac built on that state. The singular vector has the highest-weight parameters $(x',y') = (x-2a\sqrt{p}, y+a\sqrt{p})$. Similarly, if $y = b\sqrt{p} - \frac{d}{\sqrt{p}}$ with $bd > 0$, then a singular vector occurs on the level bd and has the highest-weight parameters $(x'',y'') = (x+b\sqrt{p}, y-2b\sqrt{p})$.

3.4.2. It follows that

$$\mathcal{W}(z) = e^{p\varphi_\alpha(z)+p\varphi_\beta(z)} \doteq \big|2\sqrt{p}-\tfrac{1}{\sqrt{p}}, 2\sqrt{p}-\tfrac{1}{\sqrt{p}}\big\rangle,$$

and hence the corresponding Verma-module state has two singular vectors at level 2. Both of them vanish in our free-field realization. Of the two fields $\mathcal{E}_\alpha\mathcal{W}(z)$ and $\mathcal{E}_\beta\mathcal{W}(z)$, we concentrate on the second; it lands in the module generated from

$$e^{p\varphi_\alpha(z)} \doteq \big|3\sqrt{p}-\tfrac{1}{\sqrt{p}}, -\tfrac{1}{\sqrt{p}}\big\rangle.$$

The corresponding highest-weight state in the Verma module has singular vectors at levels 3 and $p-1$. The first of these vanishes in the free-boson realization, but the second does not, yielding just the field $\mathcal{W}_\beta(z) = \mathcal{E}_\beta\mathcal{W}(z)$, as we show in Fig. 1. We note that

$$\mathcal{W}_\beta(z) = \mathcal{P}_\beta^{[p-1]}(\partial\varphi(z))e^{p\varphi_\alpha(z)}$$

with a differential polynomials in $\partial\varphi_\alpha(z)$, $\partial\varphi_\beta(z)$ in front of the exponential; here and hereafter, we indicate the degree d of a differential polynomial as $\mathcal{P}^{[d]}$.

Totally similarly,

$$\mathcal{W}_\alpha(z) = \mathcal{E}_\alpha\mathcal{W}(z) = \mathcal{P}_\alpha^{[p-1]}(\partial\varphi(z))e^{p\varphi_\beta(z)}$$

is a descendant of

$$e^{p\varphi_\beta(z)} \doteq \big|-\tfrac{1}{\sqrt{p}}, 3\sqrt{p}-\tfrac{1}{\sqrt{p}}\big\rangle.$$

The maps of $\mathcal{W}_\alpha(z)$ by $\mathcal{E}_\beta$ and of $\mathcal{W}_\beta(z)$ by $\mathcal{E}_\alpha$ are differential polynomials (not involving exponentials). They are not descendants of the unit operator, however. We have $1 \doteq \big|\sqrt{p}-\frac{1}{\sqrt{p}}, \sqrt{p}-\frac{1}{\sqrt{p}}\big\rangle$, which implies singular vectors at levels 1, 1, 4, $2p-1$, and $2p-1$. All of these vanish in the free-field realization. In each of the grades where a level-$(2p-1)$ singular vector vanishes, another state is produced as $\mathcal{E}_\alpha(e^{p\varphi_\alpha(z)})$ and

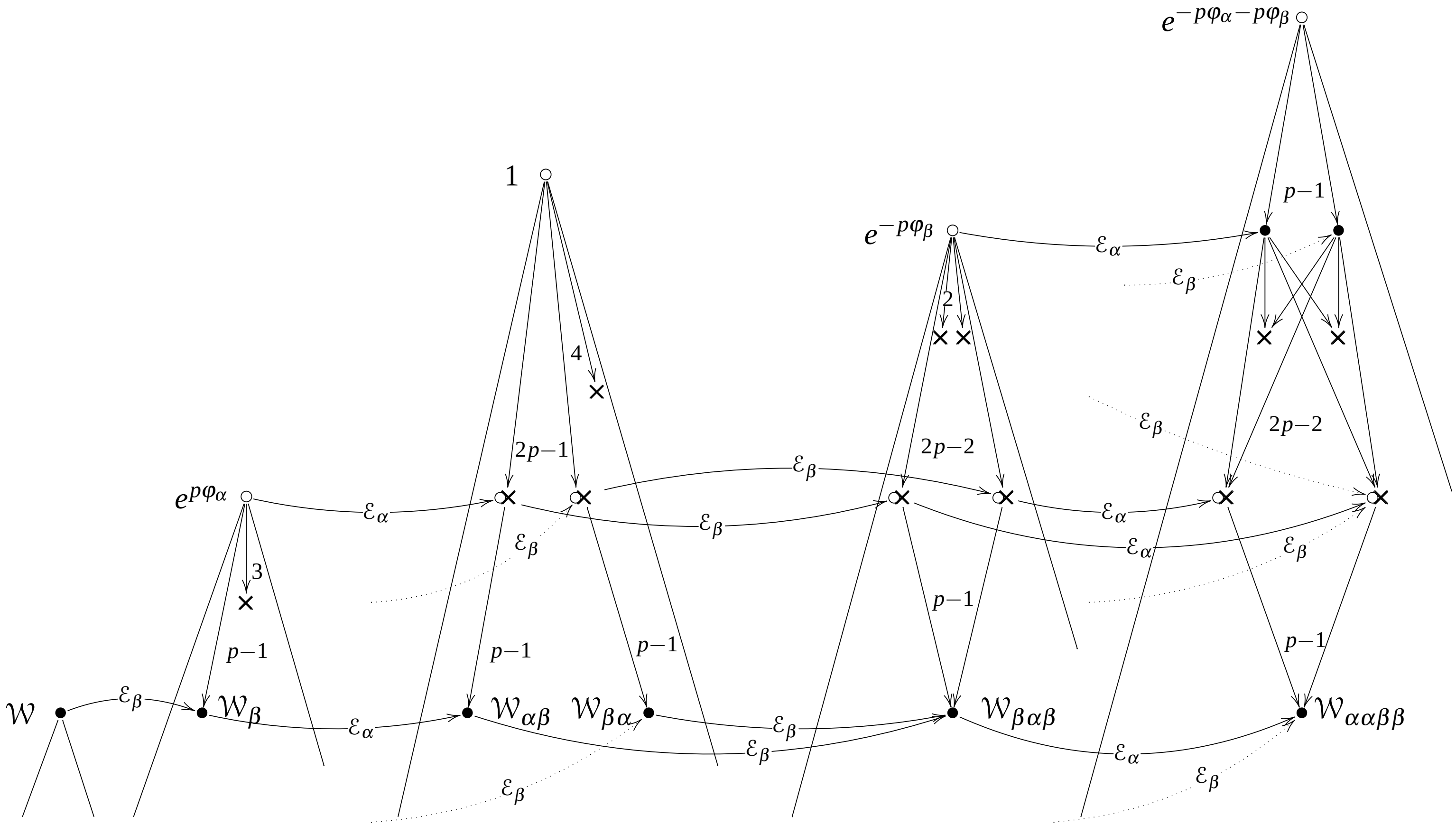

FIGURE 1. Maps by the long screenings $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$. Crosses and downward arrows leading to them show $\mathscr{W}_3$ singular vectors that vanish in the free-boson realization. Bullets (and downward arrows) show nonvanishing states in the same grades; the relative levels of singular vectors are indicated at the arrows. An open circle superimposed with a cross shows a vanishing $\mathscr{W}_3$ singular vector and a ($\mathscr{W}_3$-primary) state in the same grade, but not in the same $\mathscr{W}_3$-module (and downward arrows drawn from such $\circ\!\!\times$ show singular vectors built on those primary states). Two more modules—those with $e^{p\varphi_\beta}$ and $e^{-p\varphi_\alpha}$ at the top—are not shown here; their structure repeats that of the “$e^{p\varphi_\alpha}$” and “$e^{-p\varphi_\beta}$” modules with $\alpha \leftrightarrow \beta$. Dotted arrows show the maps by $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ *from* the missing modules.

$\mathcal{E}_\beta(e^{p\varphi_\beta(z)})$. This is shown in Fig. 1 with the $\propto$ symbols ("a state superimposed with a vanishing singular vector"). Next, $\mathcal{E}_\alpha(e^{p\varphi_\alpha(z)})$ and $\mathcal{E}_\beta(e^{p\varphi_\beta(z)})$ have singular-vector descendants on the relative level $p-1$, which are of course the respective images of $\mathcal{W}_\beta(z)$ and $\mathcal{W}_\alpha(z)$ under $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$,

$$\mathcal{W}_{\alpha\beta}(z) = \mathcal{E}_\alpha \mathcal{W}_\beta(z) = \mathcal{P}^{[3p-2]}_{\alpha\beta}(\partial\varphi(z)), \qquad \mathcal{W}_{\beta\alpha}(z) = \mathcal{E}_\beta \mathcal{W}_\alpha(z) = \mathcal{P}^{[3p-2]}_{\beta\alpha}(\partial\varphi(z)).$$

Further maps by the long screenings do not produce $\mathscr{W}_3$-descendants of the corresponding exponentials either. We consider $\mathcal{E}_\beta\mathcal{W}_{\alpha\beta}(z)$ and $\mathcal{E}_\beta\mathcal{W}_{\beta\alpha}(z)$. In the module associated with

$$e^{-p\varphi_\beta(z)} \doteq \big|2\sqrt{p} - \tfrac{1}{\sqrt{p}}, -\sqrt{p} - \tfrac{1}{\sqrt{p}}\big\rangle,$$

two singular vectors at level 2 and two at level $2p-2$ vanish; located at the grades of the last two are $\mathcal{E}_\beta\mathcal{E}_\alpha e^{p\varphi_\alpha(z)}$ (the maps shown in Fig. 1) and $\mathcal{E}_\beta\mathcal{E}_\beta e^{p\varphi_\beta(z)}$.[1] Now, $\mathcal{E}_\beta\mathcal{E}_\alpha e^{p\varphi_\alpha(z)}$ and $\mathcal{E}_\beta\mathcal{E}_\beta e^{p\varphi_\beta(z)}$ have a level-$(p-1)$ singular-vector descendant each. In our free-field realization, these two singular vectors evaluate the same up to a nonzero overall factor, thus producing a $\mathscr{W}_3$-primary field

$$\mathcal{W}_{\beta\alpha\beta}(z) = \mathcal{E}_\beta\mathcal{W}_{\alpha\beta}(z) = \mathcal{P}^{[3p-3]}_{\beta\alpha\beta}(\partial\varphi(z))e^{-p\varphi_\beta(z)}.$$

Everything with the replacement $\alpha \leftrightarrow \beta$ applies to the field

$$\mathcal{W}_{\alpha\beta\alpha}(z) = \mathcal{E}_\alpha\mathcal{W}_{\beta\alpha}(z) = \mathcal{P}^{[3p-3]}_{\alpha\beta\alpha}(\partial\varphi(z))e^{-p\varphi_\alpha(z)}.$$

Finally, mapping by the long screenings once again gives a field

$$\mathcal{W}_{\alpha\alpha\beta\beta}(z) = \mathcal{E}_\alpha\mathcal{W}_{\beta\alpha\beta}(z) = \mathcal{P}^{[4p-4]}_{\alpha\alpha\beta\beta}(\partial\varphi(z))e^{-p\varphi_\alpha(z)-p\varphi_\beta(z)}$$

(which is also $\mathcal{E}_\beta\mathcal{W}_{\alpha\beta\alpha}(z)$ up to a factor), which is not in the module associated with $e^{-p\varphi_\alpha(z)-p\varphi_\beta(z)}$, however. In the Verma module associated with the highest-weight vector

$$e^{-p\varphi_\alpha(z)-p\varphi_\beta(z)} \doteq \big|-\tfrac{1}{\sqrt{p}}, -\tfrac{1}{\sqrt{p}}\big\rangle,$$

there are two singular vectors at level $p-1$, both of which are nonvanishing in the free-field realization and are in fact the images of $e^{-p\varphi_\beta(z)}$ (and $e^{-p\varphi_\alpha(z)}$; see Fig. 1). Each of these singular vectors therefore has two level-$(2p-2)$ singular vectors, which are in fact the same pair of singular vectors. These two next-generation singular vectors vanish in

[1] We illustrate the use of **3.4.1**. In the Verma module with the highest-weight vector $\big|x,y\big\rangle = \big|2\sqrt{p} - \frac{1}{\sqrt{p}}, -\sqrt{p} - \frac{1}{\sqrt{p}}\big\rangle$ associated with $e^{-p\varphi_\beta(z)}$, one of the level-$(2p-2)$ singular vectors exists due to the representation $y = \frac{p-1}{\sqrt{p}} - 2\sqrt{p}$, and therefore the singular vector has the highest-weight parameters $(x'',y'') = (-\frac{1}{\sqrt{p}}, 3\sqrt{p} - \frac{1}{\sqrt{p}})$, i.e., those of $e^{p\varphi_\beta(z)}$. The other level-$(2p-2)$ singular vector is seen immediately if we Weyl-reflect the highest-weight parameters to $(\tilde{x},\tilde{y}) = (-x, x+y)$. We then have $\tilde{y} = \frac{2(p-1)}{\sqrt{p}} - \sqrt{p}$, and hence the singular vector has the parameters $(-3\sqrt{p} + \frac{1}{\sqrt{p}}, 3\sqrt{p} - \frac{2}{\sqrt{p}})$. After the same Weyl reflection, the parameters $(3\sqrt{p} - \frac{1}{\sqrt{p}}, -\frac{1}{\sqrt{p}})$ correspond to $e^{p\varphi_\alpha(z)}$.

our free-field realization, but the maps by $\mathcal{E}_\alpha$ (and by $\mathcal{E}_\beta$) land in the same grades. The two vectors in the image of the long screenings share a singular-vector descendant at the level-$(p-1)$ and this descendant is the $\mathcal{W}_{\alpha\alpha\beta\beta}(z)$ field.

3.4.3. We summarize the octuplet structure of $\mathscr{W}_3$ primary fields generated by long screenings from $\mathcal{W}(z)$:

$$\begin{tikzcd}
 & & 0 & & 0 & \\
 & \mathcal{W}_\beta \arrow[ur, dotted, "\mathcal{E}_\beta"] \arrow[dr, "\mathcal{E}_\alpha"] & & \mathcal{W}_{\beta\alpha\beta} \arrow[ur, dotted, "\mathcal{E}_\beta"] \arrow[dr, "\mathcal{E}_\alpha"] & & 0 \\
\mathcal{W}(z) \arrow[ur, "\mathcal{E}_\beta"] \arrow[ddr, "\mathcal{E}_\alpha"'] & & \mathcal{W}_{\alpha\beta} \arrow[ur, "\mathcal{E}_\beta"] \arrow[ddr, dashed, "\mathcal{E}_\alpha"] & & \mathcal{W}_{\alpha\alpha\beta\beta} \arrow[ur, dotted, "\mathcal{E}_\beta"] \arrow[ddr, dotted, "\mathcal{E}_\alpha"] & \\
 & & \mathcal{W}_{\beta\alpha} \arrow[uur, dashed, "\mathcal{E}_\beta"] \arrow[dr, "\mathcal{E}_\alpha"'] & & & \\
 & \mathcal{W}_\alpha \arrow[ur, "\mathcal{E}_\beta"] \arrow[dr, dotted, "\mathcal{E}_\alpha"'] & & \mathcal{W}_{\alpha\beta\alpha} \arrow[uur, "\mathcal{E}_\beta"] \arrow[dr, dotted, "\mathcal{E}_\alpha"'] & & 0 \\
 & & 0 & & 0 &
\end{tikzcd}$$

The dashed arrows represent maps to the target field up to a nonzero overall factor. All the fields in the diagram are $\mathscr{W}_3$-primaries, with the same Virasoro dimension $3p-2$.

We follow [2] in proposing these fields as generators of the octuplet algebra $\mathcal{O}_{p,1}$—the extended algebra of logarithmic $\mathscr{W}_3$ models.

3.4.4. Calculations with particular examples show the OPE

$$\mathcal{W}(z)\,\mathcal{W}_{\alpha\alpha\beta\beta}(w) = \frac{c_1\cdot 1}{(z-w)^{6p-4}} + \frac{c_2\,T(w)}{(z-w)^{6p-6}} + \frac{c_2/2\,\partial T(w)}{(z-w)^{6p-7}} + \dots$$

with *nonzero* p-dependent coefficients (and no dimension-3 $W(w)$ field), and

$$\mathcal{W}_\alpha(z)\,\mathcal{W}_{\beta\alpha\beta}(w) = \frac{(-1)^{p+1}c_1\cdot 1}{(z-w)^{6p-4}} + \frac{(-1)^{p+1}c_2\,T(w)}{(z-w)^{6p-6}} + \frac{(-1)^{p+1}c_2/2\,\partial T(w)}{(z-w)^{6p-7}} + \dots,$$
$$\mathcal{W}_\beta(z)\,\mathcal{W}_{\alpha\beta\alpha}(w) = \frac{(-1)^{p+1}c_1\cdot 1}{(z-w)^{6p-4}} + \frac{(-1)^{p+1}c_2\,T(w)}{(z-w)^{6p-6}} + \frac{(-1)^{p+1}c_2/2\,\partial T(w)}{(z-w)^{6p-7}} + \dots.$$

with *nonzero* coefficients, and the OPEs $\mathcal{W}_\alpha(z)\,\mathcal{W}_{\beta\alpha\beta}(w)$ and $\mathcal{W}_\beta(z)\,\mathcal{W}_{\alpha\beta\alpha}(w)$ that start very similarly. The adjoint-$s\ell(3)$ nature of the octuplet manifests itself in the OPEs such as

$$\mathcal{W}_\alpha(z)\,\mathcal{W}_\beta(w) = \frac{c_3\mathcal{W}(w)}{(z-w)^{3p-2}} + \dots,$$
$$\mathcal{W}_\alpha(z)\,\mathcal{W}_{\alpha\beta\alpha}(w) = \mathscr{O}(z-w),$$
$$\mathcal{W}_\beta(z)\,\mathcal{W}_{\beta\alpha\beta}(w) = \mathscr{O}(z-w),$$

$$\mathcal{W}_{\alpha\beta\alpha}(z)\,\mathcal{W}_{\beta\alpha\beta}(w)=\frac{c_3'\mathcal{W}_{\alpha\alpha\beta\beta}(w)}{(z-w)^{3p-2}}+\dots.$$

3.4.5. Some octuplet algebra representations. To construct CFT counterparts of the modules introduced in **2.3**, we first define the "fundamental weights" ω_i such that $\omega_i\cdot\alpha_j=\delta_{i,j}$:

$$\omega_1=\frac{p}{3}(2\alpha_1+\alpha_2),\qquad \omega_2=\frac{p}{3}(\alpha_1+2\alpha_2).$$

We let $\omega_\alpha(z)$ and $\omega_\beta(z)$ denote the corresponding fields:

$$\omega_\alpha(z)=\frac{p}{3}(2\varphi_\alpha(z)+\varphi_\beta(z)),\qquad \omega_\beta(z)=\frac{p}{3}(\varphi_\alpha(z)+2\varphi_\beta(z)).$$

Then the field

$$\mathcal{F}_{n_1,n_2}(z)=e^{\frac{1-n_1}{p}\omega_\alpha(z)+\frac{1-n_2}{p}\omega_\beta(z)}$$

has the same braiding with F_i as $V^{\{n_1,n_2\}}$ has in **2.3.1**. The dimension of $\mathcal{F}_{n_1,n_2}(z)$ is

$$\Delta_{n_1,n_2}=p-n_1-n_2+\frac{n_1^2+n_1n_2+n_2^2}{3p}-\frac{(p-1)^2}{p}$$

and, in fact, $\mathcal{F}_{n_1,n_2}(z)\doteq\big|x,y\big\rangle$ with (x,y) given by any pair from the Weyl orbit:

$$\begin{gathered}
\big(\sqrt{p}-\tfrac{n_1}{\sqrt{p}},\sqrt{p}-\tfrac{n_2}{\sqrt{p}}\big),\ \big(\tfrac{n_2}{\sqrt{p}}-\sqrt{p},\tfrac{n_1}{\sqrt{p}}-\sqrt{p}\big),\\
\big(\sqrt{p}-\tfrac{n_2}{\sqrt{p}},\tfrac{n_1+n_2}{\sqrt{p}}-2\sqrt{p}\big),\ \big(\tfrac{n_1+n_2}{\sqrt{p}}-2\sqrt{p},\sqrt{p}-\tfrac{n_1}{\sqrt{p}}\big),\\
\big(\tfrac{n_1}{\sqrt{p}}-\sqrt{p},-\tfrac{n_1+n_2}{\sqrt{p}}+2\sqrt{p}\big),\ \big(-\tfrac{n_1+n_2}{\sqrt{p}}+2\sqrt{p},\tfrac{n_2}{\sqrt{p}}-\sqrt{p}\big).
\end{gathered}$$

The corresponding $\mathscr{W}_3$ singular vectors vanish in the free-field realization. We propose the irreducible $\mathcal{O}_{p,1}$-modules generated from $\mathcal{F}_{n_1,n_2}(z)$ as counterparts of the corresponding simple Yetter–Drinfeld $\mathfrak{B}(X)$ modules, as a starting point to study the relation between the two representation categories.

4. Conclusions

We have outlined some details of the construction of the octuplet extended algebra $\mathcal{O}_{p,1}$ proposed in [2], and described the corresponding Nichols algebra $\mathfrak{B}(X)$ in rather explicit terms. Systematically comparing $\mathcal{O}_{p,1}$ representations with Yetter–Drinfeld $\mathfrak{B}(X)$ modules is very interesting from the perspective of whether the relation existing in the $W_{p,1}$ (triplet-algebra) case [24, 25, 28, 29] extends to the current $\mathscr{W}_3$-related octuplet setting.

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